Topological entropy for some isotropic cosmological models

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Abstract

The chaotical dynamics is studied in different Friedmann-Robertson-Walker cosmological models with scalar (inflaton) field and hydrodynamical matter. The topological entropy is calculated for some particular cases. Suggested scheme can be easily generalized for wide class of models. Different methods of calculation of topological entropy are compared.

I. INTRODUCTION

In recent years the problem of chaos in general relativity and classical cosmology has attracted great attention [1]. A special efforts has been made to resolve [2] the questions concerning the chaotical nature of the oscillatory

approach to singularity [3] or in other terms the chaotic nature of the mixmaster universe [4]. However, already simple isotropical closed Friedmann-Robertson-Walker models manifest some elements of chaotical behavior which should be taken into account for a correct construction of quantum cosmological theories [5].

The study of classical dynamics of closed isotropical cosmological model has a long history. First, it was noticed that in such models with a minimally coupled massive scalar field there is an opportunity to escape a singularity at contraction [7]. Then the periodical trajectories escaping a singularity were studied [8]. In Ref. [9] it was shown that the set of infinitely bouncing aperiodical trajectories has a fractal nature. Later this result in other terms was reproduced in our papers [10, 11, 12].

Here we would like to describe briefly the approach presented in [10]. The main idea consisted of the fact that in the closed isotropical model with a minimally coupled massive scalar field all the trajectories have the point of maximal expansion. The localization of the points of maximal expansion on the configuration plane (a, ϕ), where a is a cosmological radius, while ϕ is a scalar field, could be found analytically. Then the trajectories could be classified according to localization of their points of maximal expansion. The area of the points of maximal expansion is located inside the so-called Euclidean or "classically forbidden" region. Numerical investigation shows that this area has a quasiperiodical structure, zones corresponding to the falling to singularity are intermingled with zones in which are placed points

of maximal expansion of trajectories having the so-called “bounce” or point of minimal contraction. Then studying the substructure of these zones from the point of view of the possibility of having two bounces, one can see that this substructure repeats on the qualitative level the structure of the whole region of the possible points of maximal expansion. Continuing this procedure *ad infinitum* one can see that as the result one has the fractal set of infinitely bouncing trajectories.

The same scheme gives us an opportunity to see that there is also a set of periodical trajectories. All these periodical trajectories contains bounces intermingled with a series of oscillations of the value of the scalar field ϕ. It is important that there are no restrictions on the lengths of series of oscillations in this case. In the paper of Cornish and Shellard [5] the topological entropy was calculated for this case by the methods of symbolic dynamics and it was shown that it is positive. Using symbolic dynamics is not new for models of theoretical physics. For example, in the well-known paper [6] it is used for investigation of the chaotic repellor, generated by the scattering of a point particle from three hard discs in a plane, which one can consider as a model of the time evolution of a metastable configuration of particles. This repellor consists of the trajectories which remain confined to the scattering region as $t \to \infty$. If three hard discs with radius a are fixed in the plane at the vertices of an equilateral triangle with side R and $R > 3a$, then all the trajectories of this repellor are in one-to-one correspondence with doubly infinite sequences $...x_{-2}x_{-1}x_0x_1x_2...$ of the symbols $\{1, 2, 3\}$ with the natural

constraint $x_{n+1} \neq x_n$. The symbolic dynamics for it is very simple: in the sequences only doublets such as 11, 22 and 33 do not occur. The topological entropy of the system equals $\ln 2$ and therefore such a repellor is chaotic.

Below we reproduce the calculations of Ref. [5] and show how it is possible to generalize it for more complicated cases, but first we describe how introduction of matter or another modification of the model changes the structure of periodical trajectories. The structure of the paper is the following: in the second section we describe the properties of cosmological models under investigation, while in the third section we present the algorithm for the calculation of the topological entropy.

II. COSMOLOGICAL MODELS

First let us write down the action for the simplest cosmological model with the scalar field:

$$S = \int d^4x\sqrt{-g}\left\{\frac{m_P^2}{16\pi}(R-2\Lambda) + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right\}, \tag{2.1}$$

where m_P is the Planck mass, Λ is the cosmological constant. The equations of motion for a closed isotropic universe are

$$\frac{m_P^2}{16\pi}\left(\ddot{a} + \frac{\dot{a}^2}{2a} + \frac{1}{2a}\right) + \frac{a\dot{\phi}^2}{8} - \frac{m^2\phi^2 a}{8} - \frac{m_P^2}{8\pi}\Lambda a = 0, \tag{2.2}$$

$$\ddot{\phi} + \frac{3\dot{\phi}\dot{a}}{a} + m^2\phi = 0. \tag{2.3}$$

The first integral of motion of our system is

$$-\frac{3}{8\pi}m_P^2(\dot{a}^2+1) + \frac{a^2}{2}\left(\dot{\phi}^2 + m^2\phi^2 + \frac{m_P^2}{8\pi}\Lambda\right) = 0. \tag{2.4}$$

In the case when the cosmological constant is equal to zero, the form of the boundary of the Euclidean region is given by an equation which can be easily obtained from Eq. (2.4):

$$m^2a^2\phi^2 = \frac{3}{4\pi}m_P^2. \tag{2.5}$$

The form of this Euclidean region is shown in Fig. 1(a). In this model, investigated in many papers [8, 9, 10, 5] , there are periodical trajectories with an arbitrary number of oscillations of the scalar field.

It can be understood by using two famous asymptotics for this dynamical system. For large ϕ the slow-roll regime leads to a quasiexponential growth of the scale factor (so-called inflation), and in the end of this regime

$$a_{end} = a_0 exp(2\pi(\phi_0/M_p)^2)$$

where a_0, ϕ_0 are the initial values and a_{end} is the scale factor in the end of inflation.

For small ϕ the massive scalar field looks like dust perfect fluid but with small oscillations with the frequency equal to m. Using the Friedmann solution for a closed dustlike universe, it is possible to express the scale factor in the point of maximal expansion a_{me} through the value of the scale factor at the end of inflation a_{end}:

$$(ma_{end})^3 \sim ma_{me}$$

and the duration of the Friedmann stage

$$T = \pi a_{me}.$$

So, starting with ϕ_0, $\phi_0 >> m_P$, the universe will go through

$$N = m^3 a_{end}^3 = m^3 a_0^3 exp(6\pi(\phi_0/M_p)^2)$$

oscillations. This value can be arbitrarily large, because the value of the scalar field at the Euclidean boundary can also be arbitrarily large.

Of course, it in necessary to prove that the bounce is indeed possible for any large value a_{me}. It was done analytically by Starobinsky in [7] with the estimation of the bounce probability depending on a_{me}.

Inclusion of the positive cosmological constant investigated in [11] implies two opportunities: if Λ is small in comparison with the mass square of the scalar field m^2, the qualitative behavior is the same as in the model without cosmological constant; if the cosmological constant is of order m^2 the chaotical dynamics disappears in a jumplike manner [11].

It is interesting from a calculational point of view to consider the case of the negative cosmological constant. In this case the Euclidean region has the form presented in Fig. 1(b) and numerical calculation shows that the possible number of oscillations is restricted. With the growth of the absolute value of the cosmological constant this number is decreasing and at $|\Lambda| \sim 0.34m^2$ all the periodical trajectories disappear and dynamics become regular.

Let us include into consideration hydrodynamical matter with the equation of state

$$p = \gamma\epsilon, \tag{2.6}$$

where p is the pressure, ϵ is an energy density, and γ is the constant ($\gamma = 0$ corresponds to dust matter, $\gamma = 1/3$ corresponds to radiation, while $\gamma = 1$

describes the massless scalar field). Our analysis is valid for $\gamma > -1/3$. In this case the form of the boundary of the Euclidean region is given by equation

$$m^2a^2\phi^2 = \frac{3}{4\pi}m_P^2 - \frac{2D}{a^q}, \tag{2.7}$$

where D is the constant characterizing the quantity of the given type of matter in the universe and $q = 3(\gamma + 1) - 2$. The form of the Euclidean region described by Eq. (2.7) is shown in Fig. 1(c). One can see that in this case the Euclidean region is restricted from above. Numerical calculation shows that in this case again only a restricted number of oscillations is possible. Moreover, the structure of periodical trajectories in this case is much more complicated. Indeed, the law is the following long series of oscillations that can come after the bounce, followed by a short series of oscillations, while the behavior of the trajectory after a short series of oscillations is less restricted. The concrete laws governing the structure of trajectories depend on the parameters of the model under consideration. However, in this case also it is possible to calculate a topological entropy, which will be demonstrated below. Completing a discussion about the model with hydrodynamical matter one has to say that a large amount of matter suppresses chaotic behavior and at some critical values of the constant D periodical trajectories disappear and all the trajectories go to the singularity. For the above-mentioned cases, interesting from a physical point of view, values are the following: for the case $q = 4$ (massless scalar field), one has

$$Dm^4 > 0.0028m_P^2;$$

for $q = 2$ (radiation),

$$Dm^2 > 0.0093 m_P^2;$$

and for $q = 1$ (dust matter),

$$Dm > 0.023 m_P^2.$$

It is necessary to underline that bounces are still possible in such a situation (unless D is much bigger than a critical value), but there are no periodical trajectories, because immediately after bounce the trajectory has a point of maximal expansion and restores their travel to singularity.

Now we would like to consider the last example: the cosmological model with a complex scalar field which was studied earlier in papers [13]. Using the most natural representation of the complex scalar field

$$\phi = x \exp(i\theta), \tag{2.8}$$

where x is an absolute value of complex scalar field while θ is its phase. This phase is cyclical variable corresponding to the conserved quantity - the classical charge of the universe, which plays the role of the quasifundamental constant of the theory:

$$Q = a^3 x^2 \dot{\theta}. \tag{2.9}$$

The shape of the Euclidean region presented in Fig. 1(d) is given now by

$$m^2 a^2 x^2 = \frac{3}{4\pi} m_P^2 - \frac{Q^2}{a^4 x^2}. \tag{2.10}$$

Again, as in the preceding example at some critical value of the charge Q periodical trajectories and chaotical dynamics vanish, while at lower values of

charge there are again complicated rules governing the structure of periodical trajectories. The condition of regular behavior is given by the inequality:

$$Qm^2 > 0.056m_P^2.$$

It is important to notice that the real structure of trajectories in the case of isotropic models with scalar field and matter or with complex scalar field is more complicated because there are two types of bounces. These bounces could be called "upper" and "lower" depending on the value of the scalar field at which they take place. The value of the scalar field separating the two types of bounces correspond to a periodic trajectory with a maximal number of oscillations of the scalar field. However, in what follows we shall not discriminate between the two types. Generalization of suggested algorithms for the calculation of topological entropy for the "alphabet" including two types of bounces is straightforward.

III. CALCULATION OF THE TOPOLOGICAL ENTROPY

Now, let us calculate the topological entropy for some of the considered cases. In our situation strange repellor Ω is the set of all perpetually bouncing universes. We shall find topological entropy of the Poincare return map S with discrete time (see Ref. [14]), generated by the surface of section X consisting of the points, in which bounce or oscillation occurs: if x is the point of intersection (crossing) of the trajectory with the surface X, then Sx is the

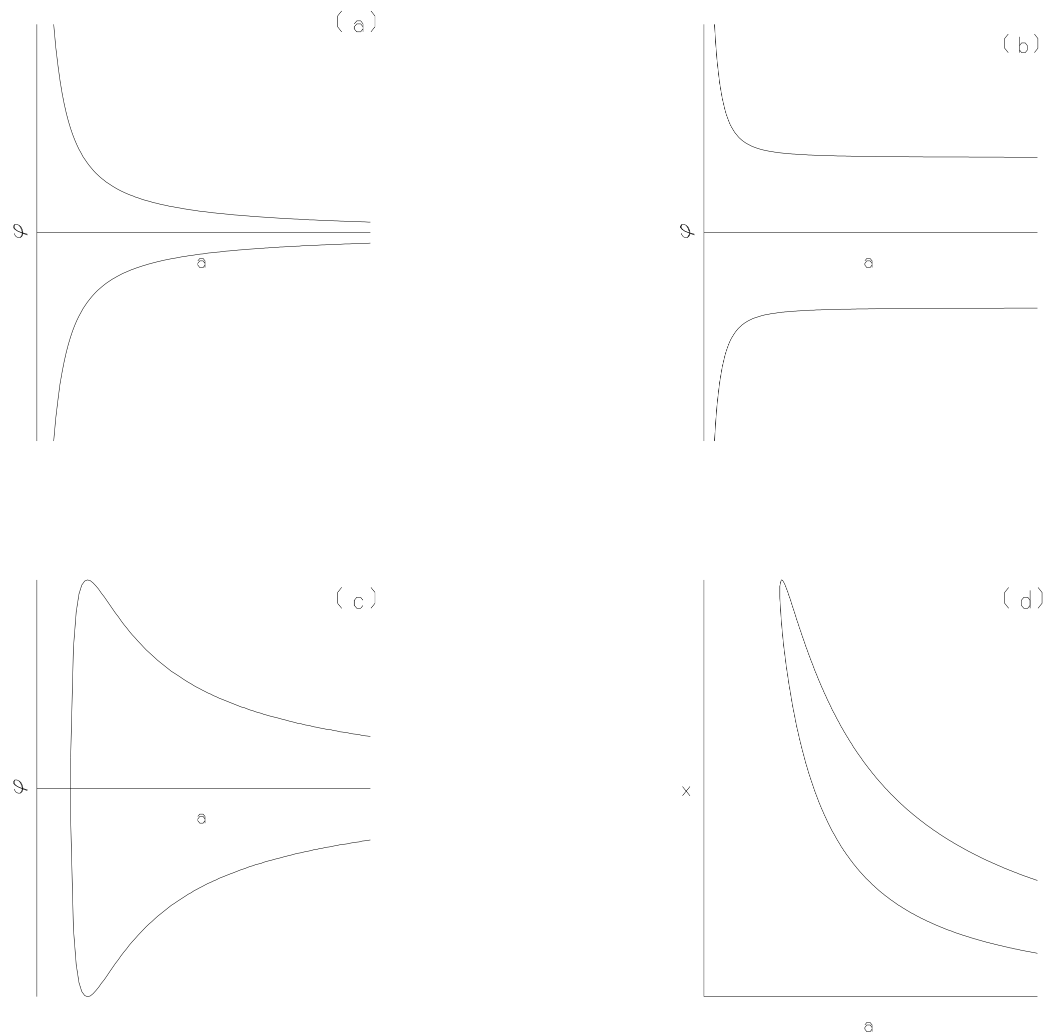

Figure 1: In Fig. 1 the shapes of Euclidean regions for different isotropic cosmological models with the scalar field are presented; a) simplest model with the scalar field, b) model with negative cosmological constant, c) model with hydrodynamical matter, d) model with complex scalar field.

point of its following intersection with X (by definition of Ω each trajectory of Ω crosses X infinitely many times). Such an entropy is independent of phase-space coordinates. We note that the topological entropy depends on the choice of Poincare surface for the section. Our Poincare surface of section X is natural for cosmological models with a massive scalar field for closed Friedmann-Robertson-Walker universes and plays the same role as the boundary of discs or obstacles for billiard systems.

Topological entropy H_T "measures" the chaos of the dynamical system. There are many equivalent definitions of topological entropy (via finite partition of the phase space) in mathematics (see Ref. [15]). But for many dynamical systems (in particular, for the finite topological Markov chain, to which the symbolic systems considered below belong) this quantity coincides with the exponent of growth in the number of periodic points as their period increases. We remark that the set of periodic points and the set of all closed orbits are everywhere dense in space X and in the repellor Ω, respectively, and thus, they contain basic information about dynamics.

More precisely, let (S, X) be a dynamical system with discrete time. If $x \in X$ and $S^k x = x$, the point x is called a periodic point of period k. We denote by $N(k)$ the number of periodic points of period or length k. Then, the definition of topological entropy is as follows:

$$H_T = \limsup_{k \to \infty} \frac{1}{k} \ln N(k). \tag{3.1}$$

If $H_T > 0$, one can conclude that the dynamics is chaotic.

One can quantify the length of the orbit by the number of symbols. To

begin with let us reproduce in some detail calculations from Ref. [5]. It was a discrete coding of orbits, using two symbols:

A, which is the bounce of the trajectory, and

B, which is the crossing of line $\phi = 0$.

It is important that for our repellor for a given admissible infinite sequence of symbols A and B there is only one trajectory, which goes through bounces and oscillations in the order defined by this sequence. Therefore, in this sequel we can deal with only such sequences.

For the simplest model with the scalar field there is the only prohibition rule: two letters A cannot stay together, which means that it is impossible to have two bounces one after another without oscillations between them (this condition is analogous to the impossibility of having two successive collisions about the same disc for the scattering from three discs). Thus our symbolic system is a topological Markov chain (of the first order) with two letters (see corresponding definition in Ref. [16]). Each topological Markov chain of the first order with k letters $\{B_1, ..., B_k\}$ is defined by topological transition matrix D. It is a $k \times k$ matrix of zeros and ones, the set of indices of which coincides with the set of letters. The entry $D_{B_1 B_2}$ of the matrix D is zero precisely when $B_1 B_2$ is a prohibited word of length 2. In our case, the topological transition matrix D is

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

(the zero in this matrix corresponds to the prohibition of the word AA). There are many practical means of counting the topological entropy. For

example, calculating the trace of the matrix D^k, we obtain the number of all period k points of the map S. Since $Tr\ D^k = \lambda_1^k + \lambda_2^k$, where λ_1 and λ_2 are eigenvalues of the matrix D, it is sufficient to find the spectrum of D. In fact, we must calculate only the largest positive eigenvalue, which exists by the Perron-Frobenius theorem (see Ref. [17]; such an eigenvalue is called Perron-Frobenius's number). In our case $\lambda_1 = \lambda_2^{-1} = (\sqrt{5}+1)/2$. The logarithm of this number is sought to define the topological entropy. Another means of calculating the topological entropy consists of finding the smallest positive root of the equation $\varphi_A(z) = 1$, where $\varphi_A(z)$ is a generating function of periodic points without intermediate recurrences in the initial symbol A (see Ref. [18]) In our case the next periodic points are of the following type: ABA, $ABBA$, ..., $ABBBBB...BBA$, ... Thus $\varphi_A(z) = z^2+z^3+...+... = z^2/(1-z)$. The smallest positive root of the equation $z^2/(1-z) = 1$ is $(\sqrt{5}-1)/2$. The logarithm of the reciprocal from this number is again a topological entropy.

Now we present the method of recurrent relations, which as one can see in sequel is more successful for our purposes. Let us denote by $Q(k)$ the number of "words" (trajectories) of length k satisfying this rule, which begin with A and end with A, and by $P(k)$ the number of words which begins with A and end with B. (We remark that our definition of coefficients $Q(k)$ and $P(k)$ is distinguished from that of Ref. [5]: in their paper $P(k)$ is the number of period k points ending in A, and $Q(k)$ is the number of period k points ending in B). Then one can easily write down the recurrent relations:

$$Q(k+1) = P(k),$$

$$P(k+1) = Q(k) + P(k). \tag{3.2}$$

It is easy to get from Eqs. (3.2) the following relation for $P(k)$:

$$P(k+1) = P(k) + P(k-1). \tag{3.3}$$

One can easily calculate that

$$P(2) = 1, \qquad P(3) = 1 \tag{3.4}$$

and that Eq. (3.3) defines the series of Fibonacci numbers. Let us recall how to find the formula for the general term of the Fibonacci series. One can look for $P(k)$ as a linear combination of terms λ^k, where λ is the solution of equation

$$\lambda^{k+1} = \lambda^k + \lambda^{k-1}$$

or, equivalently, because we are interested only in nonzero roots

$$\lambda^2 - \lambda - 1 = 0. \tag{3.5}$$

Looking for $P(k)$ in the form

$$P(k) = c_1\lambda_1^k + c_2\lambda_2^k, \tag{3.6}$$

where λ_1 and λ_2 are the roots of Eq. (3.5) and satisfying the conditions (3.4) one can get

$$P(k) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} + (-1)^{k-2}\left(\frac{\sqrt{5}-1}{2}\right)^{k-1}\right]. \tag{3.7}$$

Substituting Eq.(3.7) in the definition of the topological entropy (3.1) one can find

$$H_T = \ln\left(\frac{1+\sqrt{5}}{2}\right) > 0, \tag{3.8}$$

where $\left(\frac{1+\sqrt{5}}{2}\right)$ is the famous golden mean. It is clear that only the largest root of Eq. (3.5) is essential for the calculation of the topological entropy.

We would like to point out that the number $N(k)$ of all period k points grows as $\exp(H_T k)$ for $k \to \infty$. In our case the number $Q(k+1)$ is in fact the number of all real closed orbits (with respect to dynamical system with continuous time, generated by Einstein's equations), which are coded by k symbols (here we distinguishes closed orbit and it's multiple). In other words, $Q(k+1)$ is the number of all periodic orbits of period k (each periodic point x of period k generates periodic orbit $\{x, Sx, ..., S^{k-1}x\}$; it is clear that points $Sx, ..., S^{k-1}x$ generate the same periodic orbit; thus the number of all period k points is much greater than that of all periodic orbits of period k) and its multiples. This number grows as $\mu(\{x_0 = A\})\exp(H_T k)$ for $k \to \infty$, where μ is the measure of maximal entropy for our topological Markov chain and $\{x_0 = A\}$ is the set of all sequences, which start with the letter A (so-called "one-dimensional" cylindrical set) [19]. Such a measure is characterized by the next important property: it is the most chaotic measure among all invariant probability measures for our topological Markov chain (more precisely, it is a measure for which the Kolmogorov-Sinai entropy is equal to the topological entropy). For example, for the set of all infinite sequences of the letters A and B, such a measure is the Bernoulli measure ν

with $\nu(\{x_0 = A\}) = \nu(\{x_0 = B\}) = 1/2$. In the case of a topological Markov chain it is a Markov measure. A measure with maximal entropy plays an important role in the distribution of periodic points. There are many means of counting the probability of "one (and multi)-dimensional" cylindrical sets for such a measure. In particular, one can find the positive eigenvectors ξ and ξ^* for Perron-Frobenius's eigenvalue of the topological transition matrix A and its conjugate A^*, respectively. Taking the product of the corresponding components of the vectors ξ and ξ^* and then normalizing it, we obtain the probability of each "one-dimensional" cylindrical set. On the other hand, this number is coefficient before the k power of the largest eigenvalue in the decomposition of $Q(k+1) = P(k)$. Thus, in our case it is equal to $c_1 = (\sqrt{5}-1)/2\sqrt{5}$.

Now one can go to a more involved case of the cosmological model with scalar field and negative cosmological constant. As was described above, in this model the periodical trajectories can have only a restricted number of oscillations of the scalar field ϕ. This rule can be encoded in the prohibition to have more that n letters B staying together, where the number n depends on the parameters of this model (this dependence obtained by numerical calculations is represented in Fig. 2).

Thus we have a topological Markov chain of order $n+1$. Correspondingly, the order of the topological transition matrix $D(n)$ is equal to 2^{n+1}. We see that the first method is not well applicable to our case (in general, to any typical big matrix; this method is useful for theoretical investigations).

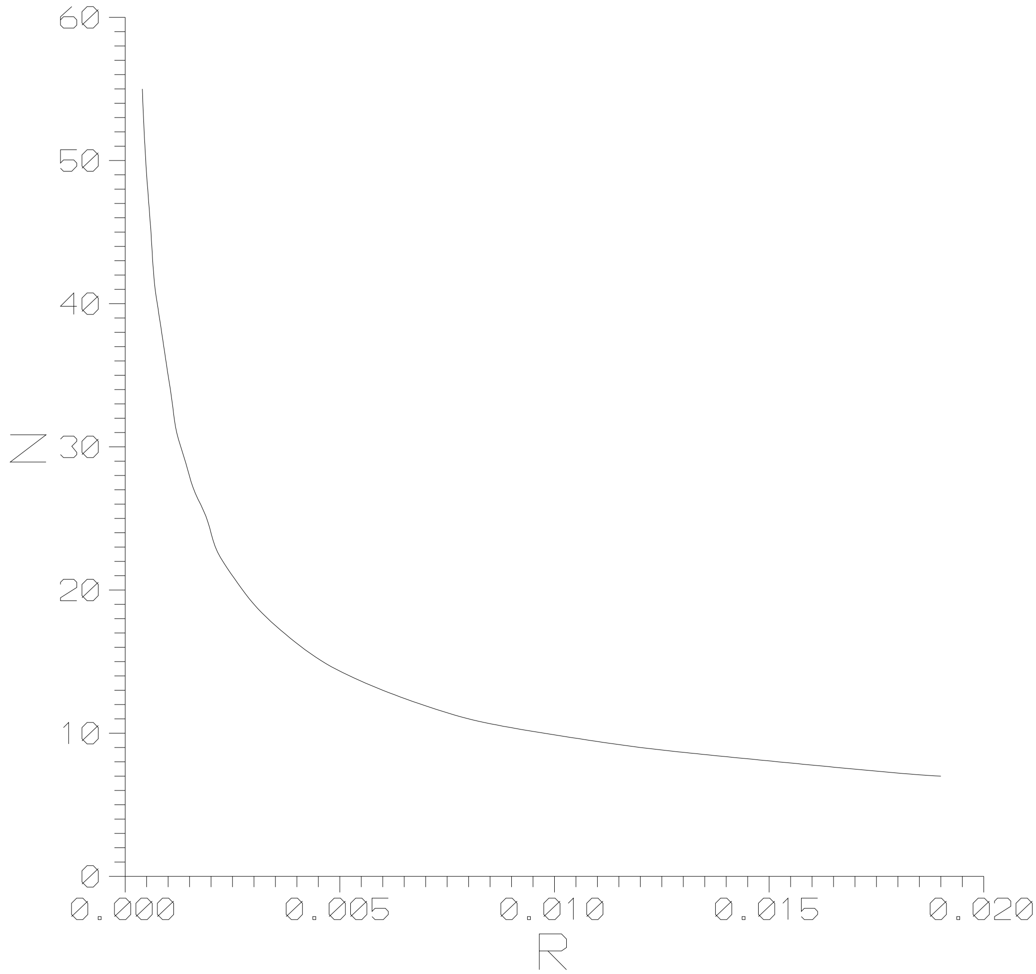

Figure 2: In Fig. 2 the dependence of the number of possible oscillations N on the ratio $R = |\Lambda|/m^2$ is represented.

Using the second method we have $\varphi_A(z) = \sum_{k=0}^{n-1} z^{2+k} = z^2(z^n - 1)/(z - 1)$ and therefore topological entropy is equal to $-\ln z(n)$, where $z(n)$ is the smallest root of the next equation $z^{n+2} - z^2 - z + 1 = 0$ (the largest positive eigenvalue $\lambda(n)$ of the matrix $D(n)$ is connected with the number $z(n)$ by the next relation: $\lambda(n) = z(n)^{-1}$).

At this time the recurrent relations is as follows:

$$\begin{aligned} Q(k+1) &= P(k), \\ P(k+1) &= Q(k) + P(k) - Q(k-n)\theta(k-n), \end{aligned} \tag{3.9}$$

where the θ - function is defined in the usual manner. We are interested in the limit $k \to \infty$ and can substitute instead of $\theta(k-n)$ number 1. Now one can write down the recurrent relation:

$$P(k+1) = P(k) + P(k-1) - P(k-n-1), \tag{3.10}$$

which in turn implies the following equation for topological entropy:

$$\lambda(n)^{n+2} - \lambda(n)^{n+1} - \lambda(n)^n + 1 = 0, \tag{3.11}$$

where topological entropy is equal to the logarithm of the biggest root of Eq. (3.11):

$$H_T = \ln \lambda(n). \tag{3.12}$$

For the small values of n, the biggest root of Eq. (3.11) could be found analytically.

For $n = 1$, we have $\lambda(1) = 1$ and topological entropy is equal to zero and chaotic behavior is absent, which is clear from physical point of view.

For $n = 2$, we obtain

$$\lambda(2) = \frac{1}{3}\left(\frac{27}{2} - \frac{3\sqrt{69}}{2}\right)^{1/3} + \frac{\left(\frac{1}{2}(9+\sqrt{69})\right)^{1/2}}{3^{2/3}} \approx 1.32. \tag{3.13}$$

For $n = 3$, one can get

$$\begin{aligned}\lambda(3) &= \frac{1}{3} + \frac{1}{3}\left(\frac{29}{2} - \frac{3\sqrt{93}}{2}\right)^{1/3} \\ &+\frac{1}{3}\left(\frac{29}{2} + \frac{3\sqrt{93}}{2}\right)^{1/2} \approx 1.47. \end{aligned} \tag{3.14}$$

For higher values of n one can find λ numerically, for example for $n = 4$, $\lambda(4) \approx 1.53$.

For large values of n one can find an asymptotical value for the biggest root λ:

$$\lambda(n) = \frac{1+\sqrt{5}}{2} - \frac{1}{2}\left(\frac{\sqrt{5}-1}{2}\right)^{n-1} + O\left(\left(\frac{\sqrt{5}-1}{2}\right)^{2n}\right). \tag{3.15}$$

Thus $\lambda(n)$ converges to golden mean exponentially fast when $n \to \infty$ (or, negative cosmological constant tends to zero).

Concerning the coefficient before k-power of the largest eigenvalue or, the probability of "one-dimensional" cylindrical set corresponding to the letter A we may show that in our case it is equal to the reciprocal of $z\frac{d}{dz}\varphi_A(z)|_{z=z(n)} = 2 + \frac{z(n)}{1-z(n)} - n\frac{z(n)^{n+2}}{1-z(n)}$ (see Ref. [18]). Since $z(n)$ tends to $(\sqrt{5}-1)/2$ exponentially fast when $n \to \infty$, we see that it is true for the convergence of $\mu_n(\{x_0 = A\})$ to $\mu(\{x_0 = A\})$, where μ_n is a measure of maximal entropy for the topological Markov chain with the topological transition matrix $D(n)$. Thus two basic parameters in the distribution of periodic

orbits of our approximating systems develop very regular behavior when the negative cosmological constant tends to zero.

As it has already been described above in the model with the scalar field and matter or in the model with the complex scalar field and nonzero classical charge, the rules governing the structure are rather complicated. Nevertheless, in this case also it is possible to calculate the topological entropy. Here we shall consider one particular example, however, the algorithm which shall be presented could be used for different sets of rules as well.

Thus, let us formulate rules for our model: (1) It is impossible to have more than 19 letters B; (2) after a series with 19 letters B (and letter A), one can have the next series only with 1 letter B; (3) after a series with 18 letters B, one can have the next series with 1 or 2 letters B; (4) after a series with 17 letters B, one can have the next series with 1, 2 or 3 letters B, $\ldots$ (20) after a series with 1 letter B, one can have the series with n letters B, where $0 \leq n \leq 19$.

These rules define the topological Markov chain of order 22. The method of calculating the topological entropy via finding the spectrum of the topological transition matrix and the method of generating functions are not applicable to this case. But the method of recurrent relations allows us to write the equation for topological entropy (we do not calculate the coefficient before the k power of the largest eigenvalue in the distribution of periodic orbits because it is a very difficult problem now).

Let us notice that the system of rules has a remarkable symmetry with re-

spect to the number $n_C = 10$, which makes further calculations more simple. These symmetrical rules give us a good approximation for the description of a real physical situation. The value n_C is apparently a function of the parameters of the model under investigation. More detailed numerical investigation implies a more complicated system of rules, however, it also could be formalized in the system of recurrent relations. Below we shall see that already the symmetric system of rules and a comparatively small number, which we have chosen for our example ($n_C = 10$) gives rather a cumbersome equation for the topological entropy.

Let us now introduce the following notation:

$Q(k)$ is the number of words which begin with letter A and end with letter A,

$Q_1(k)$ is the number of words beginning with letter A and ending with a series with 1 letter B,

$Q_2(k)$ is the number of words beginning with letter A and ending with a series with 2 letters B,

$\ldots,$

$Q_{19}(k)$ is the number of words beginning with letter A and ending with a series with 19 letters B.

The system of recurrence relations for these quantities is as follows:

$$Q(k+1) = Q_1(k) + Q_2(k) + \cdots + Q_{19}(k),$$

$$Q_1(k) = Q(k-1),$$

$$Q_{19}(k) = Q_1(k-20),$$

$$Q_d(k) = Q_{d-1}(k-1) - Q_{21-d}(k-d-1),\ 2 \leq d \leq 10$$

$$Q_d(k) = Q_{d+1}(k+1) + Q_{20-d}(k-d-1),\ 11 \leq d \leq 18 \qquad (3.16)$$

Resolving this system with respect to $Q(k)$, one get the following recurrent relation:

$$Q(k+1) = Q(k-1) + Q(k-2) + Q(k-3) + Q(k-4)$$
$$+Q(k-5) + Q(k-6) + Q(k-7) + Q(k-8) + Q(k-9)$$
$$+Q(k-10) + Q(k-13) + 2Q(k-14) + 3Q(k-15) + 4Q(k-16)$$
$$+5Q(k-17) + 6Q(k-18) + 7Q(k-19) + 8Q(k-20) + 9Q(k-21)$$
$$-9Q(k-24) - 16Q(k-25) - 21Q(k-26) - 24Q(k-27)$$
$$-25Q(k-28) - 24Q(k-29) - 21Q(k-30) - 16Q(k-31)$$
$$-9Q(k-32) - 8Q(k-36) - 21Q(k-37) - 36Q(k-38)$$
$$-50Q(k-39) - 60Q(k-40) - 63Q(k-41) - 56Q(k-42)$$
$$-36Q(k-43) + 36Q(k-47) + 84Q(k-48) + 126Q(k-49)$$
$$+150Q(k-50) + 150Q(k-51) + 126Q(k-52) + 84Q(k-53)$$
$$+36Q(k-54) + 28Q(k-59) + 84Q(k-60) + 150Q(k-61)$$
$$+200Q(k-62) + 210Q(k-63) + 168Q(k-64) + 84Q(k-65)$$
$$-84Q(k-70) - 224Q(k-71) - 350Q(k-72) - 400Q(k-73)$$
$$-350Q(k-74) - 224Q(k-75) - 84Q(k-76) - 56Q(k-82)$$
$$-175Q(k-83) - 300Q(k-84) - 350Q(k-85) - 280Q(k-86)$$
$$-126Q(k-87) + 126Q(k-93) + 350Q(k-94) + 525Q(k-95)$$

$$
\begin{aligned}
&+525Q(k-96)+350Q(k-97)+126Q(k-98)+70Q(k-105)\\
&+210Q(k-106)+315Q(k-107)+280Q(k-108)+126Q(k-109)\\
&-126Q(k-116)-336Q(k-117)-441Q(k-118)-336Q(k-119)\\
&-126Q(k-120)-56Q(k-128)-147Q(k-129)-168Q(k-130)\\
&-84Q(k-131)+84Q(k-139)+196Q(k-140)+196Q(k-141)\\
&+84Q(k-142)+28Q(k-151)+56Q(k-152)+36Q(k-153)\\
&-36Q(k-162)-64Q(k-163)-36Q(k-164)-8Q(k-174)\\
&-9Q(k-175)+9Q(k-185)+9Q(k-186)+Q(k-197)\\
&-Q(k-208), \qquad (3.17)
\end{aligned}
$$

which in turn gives the following equation for the topological entropy:

$$
\begin{aligned}
&x^{209}-x^{207}-x^{206}-x^{205}-x^{204}-x^{203}-x^{202}-x^{201}-x^{200}\\
&-x^{199}-x^{198}-x^{195}-2x^{194}-3x^{193}-4x^{192}-5x^{191}-6x^{190}\\
&-7x^{189}-8x^{188}-9x^{187}+9x^{184}+16x^{183}+21x^{182}+24x^{181}\\
&+25x^{180}+24x^{179}+21x^{178}+16x^{177}+9x^{176}+8x^{172}+21x^{171}\\
&+36x^{170}+50x^{169}+60x^{168}+63x^{167}+56x^{166}+36x^{165}-36x^{161}\\
&-84x^{160}-126x^{159}-150x^{158}-150x^{157}-126x^{156}-84x^{155}\\
&-36x^{154}-28x^{149}-84x^{148}-150x^{147}-200x^{146}-210x^{145}\\
&-168x^{144}-84x^{143}+84x^{142}+224x^{137}+350x^{136}+400x^{135}\\
&+350x^{134}+224x^{133}+84x^{132}+56x^{126}+175x^{125}+300x^{124}\\
&+350x^{123}+280x^{122}+126x^{121}-126x^{115}-350x^{114}-525x^{113}\\
&-525x^{112}-350x^{111}-126x^{110}-70x^{103}-210x^{102}-315x^{101}
\end{aligned}
$$

$$
\begin{aligned}
&-280x^{100} - 126x^{99} + 126x^{92} + 336x^{91} + 441x^{90} + 336x^{89} \\
&+126x^{88} + 56x^{80} + 147x^{79} + 168x^{78} + 84x^{77} - 84x^{69} \\
&-196x^{68} - 196x^{67} - 84x^{66} - 28x^{57} - 56x^{56} - 36x^{55} \\
&+36x^{46} + 64x^{45} + 36x^{44} + 8x^{34} + 9x^{33} - 9x^{23} \\
&-9x^{22} - x^{11} + 1 = 0. \qquad (3.18)
\end{aligned}
$$

Resolving numerically Eq. (3.18) one can find the biggest root which is equal to

$$\lambda \approx 1.61771.$$

Correspondingly the topological entropy is given by the logarithm of the biggest root. We remark that the degree (209) of our equation is much smaller than the order of the topological transition matrix (2^{22}). It means that most of the eigenvalues of the topological transition matrix are equal to zero and, therefore, this matrix contains much useless information about our dynamics. Thus we see that the method of recurrent relations is more economic in this case.

One can easily see that similar calculations can be done for every value of n_C. Here we can give the list of numbers λ corresponding to different values of n_C:

$$n_C = 2; \quad \lambda \approx 1.37747$$

$$n_C = 3; \quad \lambda \approx 1.51714$$

$$n_C = 4; \quad \lambda \approx 1.57388$$

$$n_C = 5; \quad \lambda \approx 1.59837$$

$$n_C = 10; \quad \lambda \approx 1.61771$$

$$n_C = \infty; \quad \lambda = (1+\sqrt{5})/2 \approx 1.61803.$$

Thus, the topological entropy tends to the golden mean rather rapidly.

Using this algorithm it is possible to get the equations for the topological entropy in the case of the two types of bounce (see the end of Sec.2). We will not present here the explicit forms of these equations, which differ from the previous case only in the values of nonzero coefficients, but present only some results.

First, the analog of the golden mean will be the number 2. We can see that there is not a smooth transition in the limit $D \to 0$ (when $D = 0$ we have the simplest case with only one type of bounce). But from a physical point of view, when $D \to 0$ the values of the scalar field, corresponding to the upper bounce grow, cross the Planck limit, and the upper bounce becomes impossible. So the transition to the simplest case takes place.

And finally, we present for illustration some values for the topological entropy in the symmetric case with two types of bounce:

$$n_C = 2; \quad \lambda \approx 1.82657$$

$$n_C = 3; \quad \lambda \approx 1.94561$$

$$n_C = 4; \quad \lambda \approx 1.98273$$

$$n_C = 10; \quad \lambda \approx 1.99999$$

$$n_C = \infty; \quad \lambda = 2.$$

We investigate the behavior of the topological entropy of some cosmological models, when they are close to the simple isotropical closed Friedmann-Robertson-Walker model [its parameters (negative cosmological constant, charge) tend to zero]. It was shown that chaotic behavior is robust and topological entropy converges to that of the FRW model. In the case of the presence of a negative cosmological constant, we give a full description of the corresponding symbolic dynamics and the laws of the convergence of basic parameters in the distribution of periodic orbits to that of the FRW model. The situation of the presence of matter in the model with a real scalar field and the model with a complex scalar field are more difficult, but using the method of recurrent relations we may find the values of the topological entropy. This is a case when the topological entropy is calculated for such complicated symbolic dynamics (the corresponding topological transition matrix has 209 nonzero eigenvalues). The scheme described in this example can be applied to many different physical models, obeying the different sets of prohibition rules governing the structure of periodical trajectories.

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